JOURNAL OF COMPUTATIONAL PHYSICS 9, 372–374 (1972)

Note

Intrinsic Momentum Computation of a Charged Particle in a Magnetic Field*

A charged particle injected at an angle β into a uniform magnetic field will follow a path which is a circular helix of radius a, say, and with a "dip angle" of β. The component of momentum perpendicular to B of such a particle is given by

$$p \sin \beta = cBa, \tag{1}$$

where p may be given in Bev/c, B in kilogauss, a in centimetres and where c is the speed of light.

The curvature κ, and torsion τ of a circular helix are given by [1]

$$\kappa = \frac{1}{a} \sin^2 \beta, \tag{2}$$

$$\tau = \frac{1}{a} \sin \beta \cos \beta. \tag{3}$$

Hence,

$$\kappa^2 + \tau^2 = \frac{1}{a^2} \sin^2 \beta. \tag{4}$$

The radius of the helix may thus be expressed in terms of the curvature and torsion of the curve

$$a = \frac{\kappa}{\kappa^2 + \tau^2}. \tag{5}$$

Making use of (2), we see that

$$\sin \beta = (a\kappa)^{1/2} = \frac{\kappa}{(\kappa^2 + \tau^2)^{1/2}}. \tag{6}$$

We may thus rewrite (1) as

$$p = \frac{cB}{(\kappa^2 + \tau^2)^{1/2}}. \tag{7}$$

* This work originated while the author was at Argonne National Laboratory, 1964–1965.

It may be remarked here that the expression $(\kappa^2 + \tau^2)^{1/2}$ is in general the magnitude of the screw curvature of a curve $\mathscr{C}$, i.e., the arc-rate of turning of the principal normal to the curve $\mathscr{C}$. See [1].

Letting $\bar{r}'$, $\bar{r}''$, and $\bar{r}'''$ represent the first, second and third derivatives, respectively, of the position vector $\bar{r}$ (cartesian coordinates) with respect to *arc length*, the square of the torsion is given by

$$\tau^2 = \frac{1}{\kappa^2}(\bar{r}'''^2 - \kappa^4 - \kappa'^2). \tag{8}$$

Hence, the reciprocal of the magnitude of the screw curvature of the helix is given by

$$(\kappa^2 + \tau^2)^{-1/2} = \frac{\kappa}{(r'''^2 - \kappa'^2)^{1/2}}. \tag{9}$$

Denoting the derivatives with respect to arc length of the cartesian coordinates of points on the circular helix by x', y', z', x'', y'', z'', x''', y''', and z''',

$$\bar{r}(s) = x(s)\,\bar{\imath} + y(s)\,\bar{\jmath} + z(s)\,\bar{k}, \tag{10}$$

$$\kappa^2 = \bar{r}''^2 = x''^2 + y''^2 + z''^2, \tag{11}$$

$$\kappa' = \frac{x''x''' + y''y''' + z''z'''}{(x''^2 + y''^2 + z''^2)^{1/2}}. \tag{12}$$

With only simple manipulations of (10)–(12), the right-hand side of (9) may be written as

$$\frac{\bar{r}''^2}{|\, r'' \times r''' \,|}, \tag{13}$$

and (7) may be written as

$$p = cB\frac{\bar{r}''^2}{|\, r'' \times r''' \,|}. \tag{14}$$

Cartesian x, y, z coordinates and cylindrical t, ϕ, z coordinates are related by

$$x = t\cos\phi \tag{15}$$

$$y = t\sin\phi \tag{16}$$

$$z = z. \tag{17}$$

Furthermore,

$$\frac{d\phi}{ds} = \left(\left(\frac{\partial x}{\partial\phi}\right)^2 + \left(\frac{\partial y}{\partial\phi}\right)^2 + \left(\frac{\partial z}{\partial\phi}\right)^2\right)^{-1/2}. \tag{18}$$

Suppose that the coordinates of points on a track are obtained (from stereoscopic photographs of events in a spark or bubble chamber, say) and that polynomials

$$t = t(\phi), \tag{19}$$

$$z = z(\phi), \tag{20}$$

are fitted to these points to obtain a cylindrical representation of the track. Then, using (15)–(18), it is possible to compute x', y', z', x'', etc.

Thus, if there also exists a "map" of the magnetic field $B = B(x, y, z)$, it is possible to compute (14), even when the field is nonuniform, from the curvature and torsion of the path followed by the charged particle.

It is planned to report at a later date on certain problems which may be connected with the orbital parameterization announced in this note. A detailed study shall be made of actual tracks to determine whether and how errors in track measurement affect the expression (14) for the momentum. Further, consideration shall be given to spiralling tracks involving momentum loss and to the problem of error propagation.

Acknowledgment

I should like to thank the reviewer for helpful comments on the form and content of this note. I was also helped by suggestions from and discussions with J. Butler and Richard Royston.

Received: June 5, 1970

Reference

1. C. E. Weatherburn, "Differential Geometry of Three Dimensions," Volume I, Cambridge University Press, Cambridge, England, 1931.

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